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Aspects of edge list-colourings

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Abstract

An assignment of colours to the edges of a multigraph is called an *s-improper edge-colouring* if no colour appears on more than s edges incident with any given vertex. We prove that if $L:E(G) \to 2^{\mathbb{N}}$ is an assignment of lists of colours to the edges of a multigraph G with $|L(e)| \geqslant \lceil \max\{d(u), d(v)\}/s \rceil$ for every edge e joining vertices u and v, and either s is even or G is bipartite, then G has an s-improper L-edge-colouring in which no colour appears on too many parallel edges. We prove these results using a new vertex-splitting lemma which generalizes the vertex-splitting lemma of Hilton et al. (J. Combin. Theory Ser. B 72 (1998) 91–103). We present applications of our results to school timetabling and conference scheduling. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

In a slight departure from convention, we shall allow both our graphs and multigraphs to have loops; thus, for us, a graph shall be a pair (V, E), where V and E are finite sets and $E \subseteq V^{(2)} \cup V$, and a multigraph shall be a triple (V, E, I), where V and E are finite sets and I is a mapping from E to $V^{(2)} \cup V$. Furthermore, the underlying graph of a multigraph (V, E, I) shall be the graph $(V, I(E))$. Suppose that G is a multigraph. A *list-assignment* to the edges of G is a map $L:E(G) \to 2^{\mathscr{C}}$, where $\mathscr{C}$ is a set of colours; thus, every edge e of G is assigned a subset $L(e)$ of $\mathscr{C}$. An *L-edge-colouring* of G is a map $\phi:E(G) \to \mathscr{C}$ such that $\phi(e) \in L(e)$ for every edge $e \in E(G)$. If ϕ is an L-edge-colouring such that $\phi(e_1) \neq \phi(e_2)$ whenever e_1 and e_2 are incident edges, then ϕ is called a *proper L-edge-colouring*. The least integer j such that for every list-assignment $L:E(G) \to 2^{\mathscr{C}}$ satisfying $|L(e)| \geqslant j$ for every edge $e \in E(G)$, G has a proper L-edge-colouring, is called the *choice index* of G, and is written $c'(G)$. In the case when all lists are constrained to be the same, the corresponding parameter

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is $\chi'(G)$, the chromatic index of G. The well-known list colouring conjecture is:

Conjecture A. For every loopless multigraph G we have $c'(G) = \chi'(G)$.

For a positive integer s, an *s-improper L-edge-colouring* of G is an L-edge-colouring such that for every vertex v of G and every colour $c \in \mathscr{C}$, at most s edges of G incident with v are coloured c (loops being counted twice). The least integer j such that for every list-assignment $L : E(G) \to 2^{\mathscr{C}}$ with $|L(e)| \geqslant j$ for every edge e of G, G has an s-improper L-edge-colouring, is the *s-improper choice index* of G, and is denoted $c'_s(G)$. In the case, when all lists are constrained to be the same, the corresponding parameter is denoted by $\chi'_s(G)$ and is called the *s-improper chromatic index* of G. The following conjecture is equivalent to the list-colouring conjecture:

Conjecture B (Hilton [6]). For every positive integer s and every multigraph G (G loopless if $s = 1$), we have $c'_s(G) = \chi'_s(G)$.

Clearly, if G is loopless, we have $c'(G)=c'_1(G)$ and $\chi'(G)=\chi'_1(G)$ and an 1-improper edge-colouring is a proper edge-colouring.

In 1995, Galvin [5] proved the list-colouring conjecture in the case when G is a bipartite multigraph; thus, Galvin proved that if G is a bipartite multigraph, then we have $c'(G) = \chi'(G) = \Delta(G)$, where $\Delta(G)$ is the maximum degree of G. For a shorter proof of this, see [8]. Borodin et al. [1] recently improved Galvin's result in the following way. Let $f : E(G) \to \mathbb{N}$ be a function mapping the edge-set of G into positive integers. We say that G is *f-edge-choosable* if for every list-assignment $L : E(G) \to 2^{\mathscr{C}}$ satisfying $|L(e)| \geqslant f(e)$ for every edge $e \in E(G)$, G has a proper L-edge-colouring. Borodin et al. [1] proved:

Proposition 1. *Let G be a bipartite multigraph and set $f(e) = \max\{d(u), d(v)\}$ for every edge e of G joining vertices u and v. Then G is f-edge-choosable.*

Hilton et al. [7] recently generalized Galvin's result as follows. For a multigraph G with vertex set V and a function $g : V^{(2)} \cup V \to \mathbb{N} \cup \{0\}$, we shall call a (not necessarily proper) edge-colouring of G *g-edge-bounded* if for every $\{u, v\} \in V^{(2)} \cup V$, no colour occurs on more than $g(\{u, v\})$ edges of G joining u and v (loops being counted twice). Suppose that G is a multigraph and s is a positive integer. For $\bar{e} = \{u, v\} \in V^{(2)} \cup V$ set $g_{G,s}(\bar{e}) = \lceil m_G(\bar{e})/\chi'_s(G) \rceil$, where $m_G(\bar{e})$ denotes the number of edges of G joining u to v (loops being counted twice). Hilton et al. [7] proved:

Proposition 2. *Let G be a bipartite multigraph and let s be a positive integer. Then $c'_s(G)=\chi'_s(G)=\lceil \Delta(G)/s \rceil$. Moreover, for every list-assignment $L : E(G) \to 2^{\mathscr{C}}$ satisfying $|L(e)| \geqslant \chi'_s(G)$ for every $e \in E(G)$, there exists an s-improper L-edge-colouring of G which is $g_{G,s}$-edge-bounded.*

In Section 3 of this paper we give a common generalization of Propositions 1 and 2. Suppose that G is a multigraph and s is an integer. For a function $f: E(G) \to \mathbb{N}$, we shall say that G is *s-improperly f-edge-choosable* if for every list-assignment $L: E(G) \to 2^{\mathscr{C}}$ with $|L(e)| \geqslant f(e)$ for every edge $e \in E(G)$, G has an s-improper L-edge-colouring. Clearly, G is 1-improperly f-edge-choosable if and only if G is f-edge-choosable.

Suppose that G is a multigraph and s is a positive integer. For every $\bar{e} = \{u, v\} \in V^{(2)} \cup V$ set $f_{G,s}(\bar{e}) = \lceil \max\{d(u), d(v)\}/s \rceil$, and $g'_{G,s}(\bar{e}) = \varphi(m_G(\bar{e}), \lceil d(u)/s \rceil, \lceil d(v)/s \rceil)$, where for non-negative integers m, x, y satisfying $(x = 0$ or $y = 0 \Rightarrow m = 0)$ we set

$$\varphi(m,x,y) = \begin{cases} \left\lfloor \frac{m}{\max(x,y)} \right\rfloor + \left\lfloor \frac{m}{\min(x,y)} \right\rfloor - \left\lfloor \frac{\max(x,y)}{\min(x,y)} \left\lfloor \frac{m}{\max(x,y)} \right\rfloor \right\rfloor & \\ \quad + \begin{cases} 1 & \text{if } x \nmid m \text{ and } y \nmid m \\ 0 & \text{otherwise} \end{cases} & \text{if } m \neq 0, \\ 0 & \text{if } m = 0. \end{cases}$$

Note that we always have $g_{G,s}(\bar{e}) \geqslant \lceil m(\bar{e})/f_{G,s}(\bar{e}) \rceil \geqslant \lceil m(\bar{e})/\chi'_s(G) \rceil$ and that the first inequality is an equality when $d(u) = d(v)$. For an edge e of G joining vertices u and v, set $f_{G,s}(e) = f_{G,s}(\{u, v\})$.

Theorem 3. *Let G be a bipartite multigraph and let s be a positive integer. Suppose that $L: E(G) \to 2^{\mathscr{C}}$ is a list-assignment satisfying $|L(e)| \geqslant f_{G,s}(e)$ for every edge $e \in E(G)$. Then G has an s-improper L-edge-colouring which is $g'_{G,s}$-edge-bounded.*

In the case of even s, Hilton et al. [7] proved an analogue of Proposition 2 for arbitrary multigraphs. Suppose that G is a multigraph and s is a positive even integer. For every $\bar{e} = \{u, v\} \in V^{(2)} \cup V$, set

$$g''_{G,s}(\bar{e}) = \left\lceil \frac{m(\bar{e})}{\chi'_s(G)} \right\rceil + \begin{cases} 0 & \text{if } (m(\bar{e}) - 1)/\chi'_s(G) \in [2k-1, 2k] \text{ for some integer } k, \\ 1 & \text{if } (m(\bar{e}) - 1)/\chi'_s(G) \in (2k, 2k+1) \text{ for some integer } k. \end{cases}$$

Proposition 4. *Let s be a positive even integer and let G be a multigraph. Then we have $c'_s(G) = \chi'_s(G) = \lceil \Delta(G)/s \rceil$. Moreover, for every list-assignment $L: E(G) \to 2^{\mathscr{C}}$ satisfying $|L(e)| \geqslant \chi'_s(G)$ for every $e \in E(G)$, G has an s-improper L-edge-colouring which is $g''_{G,s}$-edge-bounded.*

In Section 4 we give a generalization of Proposition 4 that is strictly analogous to the generalization of Proposition 2 given in Theorem 3.

Suppose that G is a multigraph and s is a positive even integer. For every $\bar{e} = \{u, v\} \in V^{(2)} \cup V$, set $g'''_{G,s}(\bar{e}) = \vartheta(m(\bar{e}), d(u), d(v), s)$ if $u \neq v$ and $g'''_{G,s}(\bar{e}) = \vartheta'(m(\bar{e}), d(u), s)$ if $u = v$, where for non-negative integers m, x, y and positive even integer s satisfying $(x = 0$ or $y = 0 \Rightarrow m = 0)$ we set

$$\vartheta(m,x,y,s) = \max\{\varphi(m_1, \lceil x_1/(s/2) \rceil, \lceil y_2/(s/2) \rceil) + \varphi(m_2, \lceil x_2/(s/2) \rceil, \lceil y_1/(s/2) \rceil):$$
$$m_1, m_2, x_1, x_2, y_1, y_2 \in \mathbb{Z},$$

$$\begin{aligned}
&m = m_1 + m_2,\ x = x_1 + x_2,\ \ y = y_1 + y_2,\\
&|m_1 - m_2| \leqslant 1, |x_1 - x_2| \leqslant 1, |y_1 - y_2| \leqslant 1\}\\
={}& \max\{\varphi(\lceil m/2\rceil, \lceil x_1/(s/2)\rceil, \lceil y_2/(s/2)\rceil)\\
&+ \varphi(\lfloor m/2\rfloor, \lceil x_2/(s/2)\rceil, \lceil y_1/(s/2)\rceil):\\
&(x_1,x_2) \in \{(\lceil x/2\rceil, \lfloor x/2\rfloor), (\lfloor x/2\rfloor, \lceil x/2\rceil)\},\\
&(y_1,y_2) \in \{(\lceil y/2\rceil, \lfloor y/2\rfloor), (\lfloor y/2\rfloor, \lceil y/2\rceil)\}\}
\end{aligned}$$

and for non-negative integers m, x and positive integer s satisfying $(x=0 \Rightarrow m=0)$, m even, we set

$$\vartheta'(m,x,s) = \varphi(m/2, \lceil x/s\rceil, \lceil \lfloor x/2\rfloor/(s/2)\rceil).$$

Theorem 5. *Let s be a positive even integer and let G be a multigraph. Suppose that $L: E(G) \to 2^{\mathscr{C}}$ is a list-assignment satisfying $|L(e)| \geqslant f_{G,s}(e)$ for every edge $e \in E(G)$. Then G has an s-improper L-edge-colouring which is $g'''_{G,s}$-edge-bounded.*

It is well known that an edge-colouring of a bipartite graph G can be interpreted as a school timetable. Here, one vertex set corresponds to the set of classes and the other vertex set corresponds to the set of teachers. A proper edge-colouring of G with colours $\kappa_1, \dots, \kappa_r$ (with $r \geqslant \Delta(G)$) corresponds to a teaching schedule; if an edge coloured κ_ℓ joins vertices c_i and t_j (representing the ith class and the jth teacher), then teacher t_j teaches class c_i in the ℓ-th teaching period in the week.

We show in Section 4 that Proposition 2 and Theorem 3 can be used effectively to draw up a teaching schedule that takes account of the fact that some teachers might not always be available (for example, if they were part-time) and some classes may not always be available (for example, a class might be composed of pupils on a day-release scheme from some employment). If a teacher t_j is to teach a class c_i x times in the week, then the graph G will have x edges joining c_i and t_j. The edge-boundedness can be used to ensure that these classes take place on several different days — a feature that is normally thought to be desirable in a timetable.

In Section 5 we give a similar application of Proposition 4 and Theorem 5 to a conference scheduling problem.

2. Vertex-splitting lemmas

In [7], Hilton et al. proved a novel vertex splitting lemma. Here we generalize that lemma. By a *vertex-splitting* of a graph G we shall mean a graph H obtained from G by splitting every vertex v of G into a number of vertices $v_1, \dots, v_{x(v)}$ of H, and sharing out the edges of G incident with v amongst the $v_1, \dots, v_{x(v)}$. Thus, $d_G(v) = \sum_{1 \leqslant i \leqslant x(v)} d_H(v_i)$ and the edge set of H is the same as the edge set of G.

Theorem 6. *Let G be a multigraph with no isolated vertices. For every vertex v of G let $p(v)$ be an integer satisfying $p(v) \geqslant \max_{u \in V(G)} m_G(\{v,u\})$. Then there is a vertex splitting H of G such that*

(i) *every vertex v of G splits into $\lceil d_G(v)/p(v) \rceil$ vertices of H, all but at most one of them having degree $p(v)$;*
(ii) *at least one end of every multiple edge of G is not split in H.*

The vertex-splitting lemma of [7] is the special case of Theorem 6 where all the $p(v)$ are equal. The proof is very similar.

To avoid possible confusion here, note that a loop of multiplicity m incident with a vertex v can be thought of as m loops incident with v. Condition (ii) applied to a loop of multiplicity m incident with v means that each of these m edges (some of which may still be loops) having both ends incident with the vertices, say $v_1, \dots, v_x$, into which v is split, and that one of these vertices, say v_1, is incident with x such edges and $m - x$ loops for some x.

Definition. For positive integers d, p and j with $1 \leqslant j \leqslant \lceil d/p \rceil$, with $d = xp + y$ for integers x and y with $0 \leqslant y < p$, set

$$q_j(d) = \begin{cases} p & \text{if } 1 \leqslant j \leqslant \lfloor d/p \rfloor, \\ y & \text{if } \lfloor d/p \rfloor < j \leqslant \lceil d/p \rceil. \end{cases}$$

We need the following lemma, proved in [7].

Proposition 7. *Suppose that $d, p, k, \ell, \mu_1, \dots, \mu_\ell$ are positive integers with $1 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_\ell \leqslant p$, $d = \mu_1 + \cdots + \mu_\ell$ and $k = \lceil d/p \rceil$. Suppose that $I \subseteq \{1, \dots, \ell\}$ is a set with the property that for all odd $y \in \{1, \dots, \ell\}$ at most one of $\ell - y$ and $\ell - y + 1$ belongs to I. Then there is a partition of I into subsets $I_1, \dots, I_k$, some of them possibly empty, such that* $\sum_{\alpha \in I_j} \mu_\alpha \leqslant q_j(d)$ for $1 \leqslant j \leqslant k$.

Proof of Theorem 6. Let $\bar{G}$ be the underlying graph of G. Suppose that u is a vertex of $\bar{G}$, that the edges of $\bar{G}$ incident with u are $e_1, \dots, e_\ell$, where $m_G(e_i) = \mu_i$ for $1 \leqslant i \leqslant \ell$, and $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_\ell$. A loop of multiplicity m is thought of here as two edges (or more precisely as two edge-ends) each of multiplicity m. We form a link at u between the edges e_ℓ and $e_{\ell-1}$, $e_{\ell-2}$ and $e_{\ell-3}$ and so on; there will be one unlinked edge if ℓ is odd. Once this has been done at every vertex of $\bar{G}$, the edges of $\bar{G}$ have been partitioned by the links into a number of edge-disjoint open trails and circuits, where no two open trails have a common end vertex, every open trail beginning and ending at a vertex of odd degree in $\bar{G}$. Orient every circuit in one of the two possible cyclic orders. Choose one vertex in every open trail and orient every open trail so that every edge is directed towards the vertex.

Now consider a typical vertex v whose incident edges are $e_1, \dots, e_\ell$ with multiplicities $\mu_1, \dots, \mu_\ell$, where $1 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_\ell \leqslant p(v)$, linked as described above. For every integer x with $1 \leqslant x \leqslant \lfloor \ell/2 \rfloor$, at most one of $e_{\ell-2x+1}$ and $e_{\ell-2x+2}$ is directed outwards from v. Also, $d_G(v) = \sum_{\alpha=1}^{\ell} \mu_\alpha$. Set $k(v) = \lceil d_G(v)/p(v) \rceil$. Then we split the corresponding vertex v of G into $v_1, \ldots, v_{k(v)}$. Let $I = \{ j{:}e_j$ is outwardly directed$\}$, so that, by construction, I contains at most one of the numbers $\ell - 2x + 1$, $\ell - 2x + 2$ for every integer x with $1 \leqslant x \leqslant \lfloor \ell/2 \rfloor$. Then, by Proposition 7, we can partition I into $I_1, \ldots, I_k$ so that $\sum_{\alpha \in I_j} \mu_\alpha \leqslant q_j(d_G(v))$ for $j = 1, 2, \ldots, k$. For every j, we allocate all of the edges of G corresponding to e_k with $k \in I_j$ to the new vertex v_j. When all of these outwardly directed edges have been allocated, we allocate the edges incident with v which are inwardly directed by splitting the multiple edges so that additional edges are allocated to $v_1, \ldots, v_{k(v)}$ to make the degree of every v_j exactly $q_j(d_G(v))$. Since $\sum_{j=1}^{k_i} q_j(d_G(v)) = d_G(v)$, this is possible.

We now have our splitting. $V(H)$ is the union of the new vertices $v_1, \ldots, v_{k(v)}$ obtained from every vertex v of G. Suppose that $e \in E(\bar{G})$ is directed from vertex v to vertex w, and that $m_G(e) = \mu$. Then, since e is outwardly directed from v, there is an h in the set I associated with v such that $e = e_h$; let $h \in I_j$. Now the edge e is inwardly directed at w, so if w is split into $w_1, \ldots, w_{k(w)}$, the multiple edge may be split at w according to the procedure above; let λ_s edges be allocated to the vertex w_s $(1 \leqslant s \leqslant k(w))$. Then the edge $e \in E(\bar{G})$ corresponds to edges in H of multiplicities λ_s (if $\lambda_s > 0$) from v_h to w_s $(1 \leqslant s \leqslant k(w))$. This satisfies condition (ii) of the statement of the theorem. □

Theorem 6 has the following corollary.

Corollary 8. *Let G be a multigraph with no isolated vertices and suppose that $p: V(G) \to \mathbb{N}$. Then there is a vertex-splitting H of G such that*

(i) *every vertex v of G splits into $\lceil d_G(v)/p(v) \rceil$ vertices of H, all but at most one of them having degree $p(v)$,*

(ii) *for every pair $\{v, w\}$ of distinct vertices of G, no matching in H consisting only of w–v edges of G contains more than $\varphi(m(\{v, w\}), p(w), p(v))$ edges.*

Proof. We first form a vertex-splitting J of G as follows. Let every vertex v of G be split in J into v itself and additional vertices $\{v_{w1}, \ldots, v_{wt_{vw}} : w \in V(G)\}$, where $t_{vw} = \lfloor m(\{w, v\})/p(v) \rfloor$. Allocate the edges of G to the vertices of J so that $d_J(v_{wi}) = p(v)$ and $\max_{u \in V(J)} m_J(\{v, u\}) \leqslant p(v)$ for every $v, w \in V(G)$ and $i = 1, \ldots, t_{vw}$ and so that for every pair $\{w, v\}$ of vertices of G with $d_G(w) \geqslant d_G(v)$, w is not joined to v_{wi} in J for $1 \leqslant i \leqslant r_{wv} = \lfloor t_{wv} p(w)/p(v) \rfloor$.

Set $p(v_{wi}) = p(v)$ for $v, w \in V(G)$ and $i = 1, \ldots, t_{vw}$. By Theorem 6, J has a vertex-splitting H such that every vertex $v \in V(G)$ of J splits into $\lceil d_J(v)/p(v) \rceil$ vertices of H, all but at most one of them having degree $p(v)$ in H, every vertex v_{wi} of J remains unsplit in H, and at least one end of every multiple edge of J remains unsplit in H.

Note that H is a vertex-splitting of G in which every vertex v of G is split into $\lceil d_G(v)/p(v) \rceil$ vertices of H, all but at most one such vertex having degree $p(v)$. Thus, condition (i) is satisfied.

To complete the proof of the Corollary, we now prove that condition (ii) is satisfied. Suppose that $\{v, w\}$ is a pair of vertices of G with $p(w) \geqslant p(v)$ and let M be a matching in H consisting only of w–v edges of G. There are three kinds of w–v edges in G: those of the first kind, which are not incident with w in J, those of the second kind, which join w to $\{v_{w(r_{wv}+1)}, \ldots, v_{t_{vw}}\}$ in J, and those of the third kind, which join w to v in J. Since the vertices $w_{v1}, \ldots, w_{vt_{wv}}$ of J remain unsplit in H, M contains no more than t_{wv} w–v edges of G of the first kind. Furthermore, the vertices $v_{r_{wv}+1}, \ldots, v_{t_{vw}}$ also remain unsplit in H, so M contains no more than $t_{vw} - r_{wv}$ w–v edges of G of the second kind. Now, since all the w–v edges of J remain unsplit at one end in H, M contains at most one w–v edge of G of the third kind. Finally note that when either $p(w)|m(\{w, v\})$ or $p(v)|m(\{w, v\})$, there are no w–v edges of G of the third kind, and thus M contains no w–v edges of the third kind. Thus, M contains no more than

$$\begin{aligned} &t_{wv} + t_{vw} - r_{wv} + \begin{cases} 1 & \text{if } p(w) \dagger m(\{w, v\}) \text{ and } p(v) \dagger m(\{w, v\}), \\ 0 & \text{otherwise} \end{cases} \\ &\quad = \varphi(m(\{v, w\}), p(w), p(v)) \end{aligned}$$

edges, as required. Thus, condition (ii) is satisfied and the corollary is proved. □

3. Bipartite multigraphs

In this section we prove Theorem 3.

Proof of Theorem 3. Let G be a bipartite multigraph, let s be a positive integer and suppose that $L{:}E(G) \to 2^{\mathscr{C}}$ is a list-assignment such that for every edge $e \in E(G)$, we have $|L(e)| \geqslant f_{G,s}(e)$. We construct an L-edge-colouring ϕ of G which is s-improper and $g'_{G,s}$-edge-bounded.

Without loss of generality, G has no isolated vertices. For every vertex v of G, set $p(v) = \lceil d_G(v)/s \rceil$. By Corollary 8, G has a vertex-splitting H in which every vertex v of G splits into at most s vertices of H, all of degree at most $p(v)$, such that for each pair of vertices $\bar{e} = \{w, v\}$ of G, no matching in H consisting only of w–v edges of G contains more than $\varphi(m(\bar{e}), p(w), p(v)) = h_{G,s}(\bar{e})$ edges.

For every edge e of G joining vertices v and w in G and joining v' and w' in H, we have

$$\begin{aligned} |L(e)| \geqslant f_{G,s}(e) &= \lceil \max\{d_G(u), d_G(v)\}/s \rceil \\ &= \max\{p(u), p(v)\} \geqslant \max\{d_H(u'), d_H(v')\}, \end{aligned}$$

and so, by Proposition 1, H has a proper L-edge-colouring ϕ. Since every vertex of G is split into at most s vertices of H, as an L-edge-colouring of G, ϕ is s-improper. Furthermore, since for every pair of vertices $\bar{e} = \{v, w\}$ in G, the set of v–w edges of G ϕ-coloured with any given colour forms a matching in H, no colour occurs in ϕ on more than $g'_{G,s}(\bar{e})$ v–w edges of G, and so ϕ is $g'_{G,s}$-edge-bounded as an edge-colouring of G, as required. □

4. General multigraphs with s even

In this section we prove Theorem 5. We shall require a simple lemma.

Lemma 9. *Any multigraph G has an orientation D such that* (i) *for every vertex v of D, we have $|d_D^+(v) - d_D^-(v)| \leqslant 1$, and* (ii) *for every pair $\{u, v\}$ of vertices of D, we have $|m_D(u, v) - m_D(v, u)| \leqslant 1$.*

Here, $d_D^+(v)$, $d_D^-(v)$ and $m_D(u, v)$ denote the out-degree of v in D, the in-degree of v in D, and the number of arcs joining u to v in D, respectively.

Proof. It suffices to prove the lemma in the case when G is a loopless graph. We can reduce the multigraph case to the loopless graph case by removing from G pairs of parallel edges oriented in opposite directions, and by removing from G oriented loops.

So suppose that G is a loopless graph. Since G is a loopless graph, condition (ii) is satisfied by any orientation D of G. It remains to prove, therefore, that G has an orientation D satisfying condition (i).

Let $v_1, \ldots, v_{2r}$ be the vertices of odd degree in G. Let G^+ be the multigraph obtained by adding to G an edge joining vertices v_{2i-1} and v_{2i} for every $i = 1, \ldots, r$. Let D^+ be an Eulerian orientation of G^+ and let D be D^+ restricted to G. Since D differs from an Eulerian digraph by a matching, D clearly satisfies condition (i). $\square$

Proof of Theorem 5. Let G be a multigraph, let s be a positive even integer and suppose that $L{:}E(G) \to 2^{\mathscr{C}}$ is a list-assignment such that for every edge $e \in E(G)$ joining vertices u and v, we have $|L(e)| \geqslant f_{G,s}(e)$. We construct an L-edge-colouring ϕ of G which is s-improper and $g'''_{G,s}$-edge-bounded.

By Lemma 9, G has an orientation D such that for every vertex v of G we have $|d_D^+(v) - d_D^-(v)| \leqslant 1$ and for every pair of vertices $\{u, v\}$ we have $|m_D(u, v) - m_D(v, u)| \leqslant 1$.

Suppose that G has the vertex set $\{v_1, \ldots, v_p\}$. Let B be the bipartite multigraph on vertex classes $U = \{u_1, \ldots, u_p\}$ and $W = \{w_1, \ldots, w_p\}$ and edge set $E(B) = E(G)$ where an edge e of G joining v_i to v_j joins u_i to w_j in B if e is oriented from v_i to v_j in D and joins u_j to w_i in B otherwise.

We observe the following simple properties of B:

(i) $m_B(\{u_i, w_j\}) + m_B(\{u_j, w_i\}) = m_G(\{v_i, v_j\})$ for $1 \leqslant i, j \leqslant p$.
(ii) $d_B(u_i) + d_B(w_i) = d_G(v_i)$ for $i = 1, \ldots, p$,
(iii) $|m_B(\{u_i, w_j\}) - m_B(\{u_j, w_i\})| \leqslant 1$ for $1 \leqslant i, j \leqslant p$,
(iv) $|d_B(u_i) - d_B(w_i)| \leqslant 1$ for $i = 1, \ldots, p$.

If e is an edge of B joining vertices u_i and w_j, then

$$\begin{aligned} |L(e)| &\geqslant \max\{\lceil d_G(v_i)/s \rceil, \lceil d_G(v_j)/s \rceil\} \\ &= \max\{\lceil \lceil d_G(v_i)/2 \rceil/(s/2) \rceil, \lceil \lceil d_G(v_j)/2 \rceil/(s/2) \rceil\} \\ &\geqslant \max\{\lceil d_B(u_i)/(s/2) \rceil, \lceil d_B(w_j)/(s/2) \rceil\}. \end{aligned}$$

Therefore, by Theorem 3, B has a $(s/2)$-improper L-edge-colouring ϕ which is $g'_{B,s/2}$-edge-bounded. As an L-edge-colouring of G, ϕ is clearly s-improper.

It remains to show that, as an edge-colouring of G, ϕ is $g'''_{G,s}$-edge-bounded. Let $\{v_i, v_j\}$ be a pair of vertices of G. Suppose first that $i \neq j$. Since ϕ is $g'_{B,s/2}$-edge-bounded as an edge-colouring of G, no colour appears more than

$$\begin{aligned}
&g'_{B,s/2}(\{u_i, w_j\}) + g'_{B,s/2}(\{u_j, w_i\})\\
&\quad= \varphi(m(\{u_i, w_j\}), \lceil d_B(u_i)/(s/2)\rceil, \lceil d_B(w_j)/(s/2)\rceil)\\
&\qquad+ \varphi(m(\{u_j, w_i\}), \lceil d_B(u_j)/(s/2)\rceil, \lceil d_B(w_i)/(s/2)\rceil)\\
&\quad\leqslant \vartheta(m(\{v_i, v_j\}), d_G(v_i), d_G(v_j), s)\\
&\quad= g'''_{G,s}(\{v_i, v_j\})
\end{aligned}$$

times on edges of G joining v_i and v_j. Finally, if $i = j$, no colour appears more than

$$\begin{aligned}
&g'_{B,s/2}(\{u_i, w_i\})\\
&\quad= \varphi(m(\{u_i, w_i\}), \lceil d_B(u_i)/(s/2)\rceil, \lceil d_B(w_i)/(s/2)\rceil)\\
&\quad= \varphi(m(\{v_i\})/2, \lceil\lceil d_G(v_i)/2\rceil/(s/2)\rceil, \lceil\lfloor d_G(v_i)/2\rfloor/(s/2)\rceil)\\
&\quad= \vartheta'(m(\{v_i\}), d_G(v_i), s)\\
&\quad= g'''_{G,s}(\{v_i\})
\end{aligned}$$

times on loops on v_i. Thus, ϕ is indeed $g'''_{G,s}$-edge-bounded as an edge-colouring of G, as required. □

5. Applications of edge list-colourings

Here we give two possible applications of the theorems in this paper to scheduling problems.

5.1. School timetabling

It is well known that a school timetable can be thought as a properly edge-coloured bipartite multigraph G. Here, one set of vertices $\{c_1, \ldots, c_p\}$ represents the classes that have to be taught, and the other set of vertices $\{t_1, \ldots, t_q\}$ represents the teachers. A class c_i is joined to a teacher t_j by x edges if the teacher t_j teaches the class c_i x times during the week. The hours in the week when lessons take place are denoted by $h_1, \ldots, h_r$; these are thought of as colours, and so one edge joining vertex t_j to vertex c_i is coloured h_k if teacher t_j teaches class c_i in the kth hour. This edge-colouring is proper (if we assume that no class is taught by two teachers at the same time, and no teacher teaches two classes at once). This representation of a school timetable ignores the important question of which rooms the classes are to take place in.

It is generally held that too many prerequisites and conditions can make a timetable impossibly difficult to find. We indicate here how the idea of list-colouring (both proper and improper) can be used to cope with many such awkward conditions in a satisfactory and easy manner. This has been discussed in a number of recent papers (e.g. [2–4]). Here we bring out a new angle, showing how improper and proper edge list-colourings can be used together in producing a timetable, and showing how edge-boundedness can be used to ensure that the times when a particular teacher meets a particular class can be spread out over a number of days.

We suppose that it is known how many times every teacher is to meet his or her classes in the week. Thus, we suppose that the bipartite multigraph G is given. The construction of the timetable will correspond to the assignment of colours (hours) $h_1,\dots,h_r$ to the edges of G so as to obtain a proper edge-colouring.

Suppose that some of the teachers are not available every morning and every afternoon during the week. For example, some may be employed part-time, some may have regular non-teaching duties within the school, and some may wish all their free periods to be concentrated in one afternoon, so that they can learn more about quality assurance. Similarly, it could happen that some classes are not available every morning and every afternoon. For example, some classes may be composed of pupils on some day-release scheme from some factory. There can be constraints that prohibit some particular teacher from meeting some particular class at certain times. For example, a chemistry teacher might need to teach a certain class in a certain laboratory, and the laboratory might not always be available (perhaps it is occupied by another class at certain times). All such constraints can be met by assigning lists to the edges of G so that the list $L(e)$ assigned to an edge e consists of all those hours (colours) h_k when the class c_i and the teacher t_j joined by e are both available and it is possible for teacher t_j to teach class c_i. The theorem of Borodin et al. (Proposition 1) then says that, provided the list $L(e)$ is at least as large as the number of lessons given by the teacher t_j, and the number of lessons taken by the class c_i, a proper L-edge-colouring exists, that is, a suitable timetable can be constructed. Of course, a proper L-edge-colouring might exist even if the Borodin et al. condition is not satisfied.

Another feature of a timetable that is often required is that, when a teacher teaches a class several times in a week, the meetings should be spaced out. Thus, a Latin teacher would not wish to teach the Fourth Form Latin all Monday morning, and have no other classes with them in the rest of the week; apart from his natural wish to retain his sanity, less Latin would be absorbed than if the classes could be arranged to take place on different days.

A way of drawing up a timetable to prevent such catastrophes and to take into account the various possible non-availabilities of teachers and classes is to draw up the timetable in two stages. We could first divide up the week into, say, 10 sessions (corresponding to the mornings and afternoons of the days of the week from Monday to Friday). Suppose there were r teaching hours in the morning and r in the afternoon (probably $r=3$ or 4). We could assign lists to every edge corresponding to the availability of the teachers and classes in the 10 sessions. For example, if teacher t_j had to meet class c_i three times in the week, but the only sessions when this is possible are Monday afternoon, Tuesday morning and afternoon, Wednesday morning and Friday morning, then the three edges joining c_i and t_j would each have the list M_2, T_1, T_2, W_1 and F_1. We would then try to find an appropriate edge-bounded L-edge-colouring of G with 10 colours. Theorem 3 gives a sufficient condition for this to be possible. The edge-boundedness would ensure that the meetings between any particular teacher t_j and class c_i were spread out as much as possible into different sessions.

Having done this, we would look at all the 10 submultigraphs, say $G_{M_1}, G_{M_2}, \ldots, G_{F_2}$, induced by the edges of each of the 10 colours above. Taking G_{M_1}, for example, we would need to find a proper edge-colouring with r colours, $h_1, \ldots, h_r$. Further non-availability constraints could be incorporated at this stage. For example, if teacher t_j is engaged in some administrative task in the first hour, the edges incident with him or her would have lists excluding colour h_1. Proposition 1 gives a sufficient condition for this to be possible. When this is done for all the submultigraphs, we have the desired timetable.

5.2. Conference scheduling

This application is perhaps more fanciful than that of school timetabling, but it has been discussed in print.

The object is to find a schedule at a conference so that any two people who wish to have a brief (five to ten minute) conversation with each other may do so. All spontaneity is of course lost from the timing of these conversations, but one hopes that the conversations may nevertheless be worthwhile; at least all pairs of participants that wish to speak to each other and are on the schedule can do so.

Initially, having determined who wishes to talk to whom, and for how many times, a multigraph G may be constructed. The vertices $v_1, \ldots, v_n$ represent the people who are to have the conversations, and if v_i and v_j are to have x conversations, then v_i and v_j will be joined by x edges.

If there are no constraints, then the obvious course of action is to find a proper edge-colouring of G with colours $t_1, \ldots, t_r$ so that, if an edge joining vertices v_i and v_j is coloured t_k, then the conference participants v_i and v_j hold a conversation at the t_kth time slot. For such an edge-colouring to exist, it is necessary and sufficient that $r \geqslant \chi'(G)$.

Suppose now that there are constraints of the form that certain people will not be available at certain times. Then we can make use of the following result of Borodin et al. [1].

Proposition 10. *Let G be a multigraph and let $L: E(G) \to 2^{\mathscr{C}}$ be a list-assignment. If for every edge e of G joining vertices u and v we have*

$$|L(e)| \geqslant \max\{d(u), d(v)\} + \lfloor \min\{d(u), d(v)\}/2 \rfloor,$$

then G has a proper L-edge-colouring.

We assign to every edge e joining vertices v_i and v_j a list $L(e)$ of the times t_k when the conference participants v_i and v_j are available. Proposition 10 gives a sufficient condition for G to have a proper L-edge-colouring. If G has such a proper L-edge-colouring, we can use it to draw up our schedule as follows: if an edge e joining v_i and v_j is coloured t_k, then the conference participants v_i and v_j have a conversation in the t_kth time slot.

If the conference lasts several days and the participants v_i and v_j are to have several conversations, they might well prefer to have them spaced out as much as possible. Suppose the conference lasts for q days and that the same number of meetings will be arranged in every morning and afternoon session. Then we could initially assign to every edge joining v_i and v_j a list consisting of the sessions when v_i and v_j are both available. We could then look for an appropriately edge-bounded $(2q)$-improper L-edge-colouring of G. A sufficient condition for this is given in Theorem 5. Here every colour would correspond to the set of meetings to be held in a particular session; thus, for example, colour M_1 could correspond to all the meetings to be held on Monday morning. Next consider the subgraphs $G_{M_1}, G_{M_2}, G_{T_1}, \ldots$ induced by the edges of the different colours $M_1, M_2, T_1, T_2, W_1, \ldots$. For the subgraph G_{M_1}, for example, give every edge e joining, say, v_i and v_j, a list $L_{M_1}(e)$ corresponding to the times on Monday morning when they will both be available. Then give G_{M_1} a proper L_{M_1}-edge-colouring. Proposition 10 gives a sufficient condition for such a colouring to exist. As before, if an edge e joining v_i and v_j is coloured t_k in the graph G_{M_1}, then in the t_kth time slot on Monday morning, the participants v_i and v_j will converse. The edge-boundedness of the $(2q)$-improper L-edge-colouring will ensure that, as far as possible, the conversations between any pair v_i and v_j are in different sessions.

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